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SOME INEQUALITIES FOR GAUSSIAN PROCESSES AND APPLICATIONS

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ABSTRACT

We present a generalization of Slepian's lemma and Fernique's theorem. We show how these can be easily applied to give a new proof, with improved estimates, of Dvoretzky's theorem on the existence of "almost" spherical sections for arbitrary convex bodies in R^N, while avoiding the isoperimetric inequality.

Introduction

Let (Ω, F, P) be a probability space and $\{X_{i,j}\}$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$ be a doubly indexed sequence of real valued centered Gaussian r.v.s. on (Ω, F, P).

We are interested in comparing $P(\bigcap_{i=1}^{n} \bigcup_{j=1}^{m} [X_{i,j} \geqq \lambda_{i,j}])$ and $E(\min_i \max_j X_{i,j})$ with the respective analogous forms obtained from another sequence $\{Y_{i,j}\}$ of real valued centered Gaussian r.v.s. The main results in this direction are Theorems 1.1 and 1.4, which extend the well-known Slepian's lemma [8] and Fernique's Theorem [4] (see also [6]).

We shall show that Theorem 1.4 can be applied, for example, to give a new proof of the famous Dvoretzky's theorem [2] on the existence of "almost" spherical sections for arbitrary convex bodies in R^N, as well as some new estimates which are useful in the context of the study of the local structure of finite-dimensional Banach spaces.

§1. Some inequalities for Gaussian processes

The next theorem is an extension of Slepian's lemma [8], [6].

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THEOREM 1.1. *Let $\{X_{i,j}\}$ and $\{Y_{i,j}\}$ $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$ be two sequences of real valued centered Gaussian r.v.s. satisfying:*

(1) $E(X_{i,j}^2) = E(Y_{i,j}^2)$ *for all* $1 \leqq i \leqq n$, $1 \leqq j \leqq m$,

(2) $E(X_{i,j}X_{i,k}) \leqq E(Y_{i,j}Y_{i,k})$ *for all* $1 \leqq i \leqq n$, $1 \leqq j, k \leqq m$,

(3) $E(X_{i,j}X_{l,k}) \geqq E(Y_{i,j}Y_{l,k})$ *for all* $i \neq l$, $1 \leqq i, l \leqq n$, $1 \leqq j, k \leqq m$.

Then,

$$P\Big(\bigcap_{i=1}^{n} \bigcup_{j=1}^{m} [X_{i,j} \geqq \lambda_{i,j}]\Big) \geqq P\Big(\bigcap_{i=1}^{n} \bigcup_{j=1}^{m} [Y_{i,j} \geqq \lambda_{i,j}]\Big)$$

for all real scalars $\lambda_{i,j}$.

The proof of Theorem 1.1 will use the following simple lemma whose proof is omitted (A^c denotes the complement of the set A).

LEMMA 1.2. *Let $A_{i,j}$ $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$ be subsets of a given set A. Let $B_{i,0} = A_{i,1}$ and $B_{i,j} = A_{i,1}^c \cap \cdots \cap A_{i,j}^c \cap A_{i,j+1}$ for all $1 \leqq i \leqq n$, $1 \leqq j < m$. Then,*

$$\bigcap_{i=1}^{n} \bigcup_{j=1}^{m} A_{i,j} = \bigcup_{j_1=0}^{m-1} \cdots \bigcup_{j_n=0}^{m-1} (B_{1,j_1} \cap B_{2,j_2} \cap \cdots \cap B_{n,j_n}).$$

REMARK. Note that the sets $B_{1,j_1} \cap \cdots \cap B_{n,j_n}$ are distinct for distinct vectors $(j_1, \ldots, j_n)$.

PROOF OF THEOREM 1.1. We shall adopt the following notation: A vector $x = (x_1, \ldots, x_{nm})$ in R^{nm} will also be denoted by

$$x = (x_{1,1}, \ldots, x_{1,m}, x_{2,1}, \ldots, x_{2,m}, \ldots, x_{n,1}, \ldots, x_{n,m})$$

where $x_{i,j} = x_{(i-1)m+j}$ $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$.

Given any positive definite matrix $\Gamma = (\gamma_{\alpha,\beta})$, $1 \leqq \alpha, \beta \leqq nm$, let $Z = (Z_\alpha)$ be the centered Gaussian variable determined by Γ with density function

$$g(z;\Gamma) = (2\pi)^{-nm} \int_{R^{nm}} \exp\{i(x,z) - \tfrac{1}{2}(\Gamma(x), x)\} dx.$$

It is very easy to see that if $\alpha \neq \beta$ then $\partial g / \partial \gamma_{\alpha,\beta} = \partial^2 g / \partial z_\alpha \partial z_\beta$. Notice that if $\alpha = (i-1)m + j$, $\beta = (l-1)m + k$ $(1 \leqq i, l \leqq n, 1 \leqq j, k \leqq m)$ then by our notation

$$\gamma_{\alpha,\beta} = E(Z_\alpha Z_\beta) = E(Z_{i,j} Z_{l,k}).$$

Let now $A_{i,j} = [Z_{i,j} \geqq \lambda_{i,j}]$. We have by Lemma 1.2 and the remark which followed that

$$Q(Z;\Gamma)=P\left(\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}A_{i,j}\right)=\sum_{j_1=0}^{m-1}\cdots\sum_{j_n=0}^{m-1}\int_{B_{n,j_n}}\cdots\int_{B_{1,j_1}}g(z)dz$$

where for $1\leqq i\leqq n$, $0\leqq j\leqq m-1$

$$\int_{B_{i,0}}f(z)dz_{i,1}dz_{i,2}\cdots dz_{i,m}=\int_{\lambda_{i,1}}^{\infty}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{m-1}f(z)dz_{i,m}\cdots dz_{i,2}dz_{i,1}$$

for any function $f(z_{1,1},\ldots,z_{n,m})$ and

$$\int_{B_{i,j}}f(z)dz_{i,1}dz_{i,2}\cdots dz_{i,m}=\int_{-\infty}^{\lambda_{i,1}}\cdots\int_{-\infty}^{\lambda_{i,j}}\int_{\lambda_{i,j+1}}^{\infty}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{m-j-1}f(z)dz_{i,m}\cdots dz_{i,2}dz_{i,1}.$$

By differentiating Q with respect to $\gamma_{\alpha,\beta}$ we obtain that

$$\frac{\partial Q}{\partial\gamma_{\alpha,\beta}}(Z;\Gamma)=\sum_{j_1=0}^{m-1}\cdots\sum_{j_n=0}^{m-1}\int_{B_{1,j_1}}\cdots\int_{B_{n,j_n}}\frac{\partial^2 g(z)}{\partial z_\alpha\partial z_\beta}dz.$$

We shall compute $\int_{B_{1,j_1}}\cdots\int_{B_{n,j_n}}(\partial^2 g(z)/\partial z_\alpha\partial z_\beta)dz$ for all $\alpha\neq\beta$. There are two possibilities:

(a) $\alpha=(i-1)m+k$, $\beta=(i-1)m+l$, where $1\leqq k<l\leqq m$, $1\leqq i\leqq n$,

(b) $\alpha=(i-1)m+k$, $\beta=(i_0-1)m+l$, where $1\leqq k$, $l\leqq m$, $1\leqq i<i_0\leqq n$.

In case (a), without loss of generality we take $z_\alpha=z_{1,m-1}$ and $z_\beta=z_{1,m}$ (i.e., $i=1, k=m-1$, $l=m$), then

$$\int_{B_{1,j_1}}\frac{\partial^2 g(z)}{\partial z_{1,m-1}\partial z_{1,m}}dz_{1,1}\cdots dz_{1,m}$$

$$=\int_{-\infty}^{\lambda_{1,1}}\cdots\int_{-\infty}^{\lambda_{1,j_1}}\int_{\lambda_{1,j_1+1}}^{\infty}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{m-j_1-1}\frac{\partial^2 g(z)}{\partial z_{1,m-1}\partial z_{1,m}}dz_{1,m}\cdots dz_{1,1}$$

and we see that this is equal to zero if $j_1<m-1$ because the first integral with respect to $z_{1,m}$ is

$$\int_{-\infty}^{\infty}\frac{\partial^2 g(z)}{\partial z_{1,m-1}\partial z_{1,m}}dz_{1,m}=0.$$

But when $j_1 = m-1$, then

$$\int_{B_{1,j_1}} \frac{\partial^2 g(z)}{\partial z_{1,m-1}\partial z_{1,m}}\, dz_{1,1}\cdots dz_{1,m}$$
$$= -\int_{-\infty}^{\lambda_{1,1}}\cdots\int_{-\infty}^{\lambda_{1,m-2}} g(z)\Big|_{\substack{z_{1,m-1}=\lambda_{1,m-1}\\ z_{1,m}=\lambda_{1,m}}} dz_{1,m-2}\cdots dz_{1,1}.$$

Hence, it follows that in case (a), $\partial Q/\partial\gamma_{\alpha,\beta} \leqq 0$.

In case (b), without loss of generality we take $z_\alpha = z_{1,m}$ and $z_\beta = z_{2,m}$. Then, when either j_1 or j_2 is smaller than $m-1$, we obtain as above that

$$\int_{B_{2,j_2}}\int_{B_{1,j_1}} \frac{\partial^2 g(z)}{\partial z_{1,m}\partial z_{2,m}}\, dz_{1,1}\cdots dz_{1,m} dz_{2,1}\cdots dz_{2,m} = 0.$$

However, if $j_1 = j_2 = m-1$, then

$$\int_{B_{2,m-1}}\int_{B_{1,m-1}} \frac{\partial^2 g(z)}{\partial z_{1,m}\partial z_{2,m}}\, dz_{1,1}\cdots dz_{1,m} dz_{2,1}\cdots dz_{2,m}$$
$$= \int_{-\infty}^{\lambda_{1,1}}\cdots\int_{-\infty}^{\lambda_{1,m-1}}\int_{\lambda_{1,m}}^{\infty}\int_{-\infty}^{\lambda_{2,1}}\cdots\int_{-\infty}^{\lambda_{2,m-1}}\int_{\lambda_{2,m}}^{\infty} \frac{\partial^2 g(z)}{\partial z_{1,m}\partial z_{2,m}}\, dz_{2,m}\cdots dz_{1,1}$$
$$= \int_{-\infty}^{\lambda_{1,1}}\cdots\int_{-\infty}^{\lambda_{1,m-1}}\int_{-\infty}^{\lambda_{2,1}}\cdots\int_{-\infty}^{\lambda_{2,m-1}} g(z)\Big|_{\substack{z_{1,m}=\lambda_{1,m}\\ z_{2,m}=\lambda_{2,m}}} dz_{2,m-1}\cdots dz_{2,1}dz_{1,m-1}\cdots dz_{1,1}$$
$$\geqq 0.$$

Hence, it follows that in case (b), $\partial Q/\partial\gamma_{\alpha,\beta} \geqq 0$.

Let now Γ_X and Γ_Y be the covariance matrices of

$$X = (X_{1,1},\ldots,X_{1,m},\ldots,X_{n,1},\ldots,X_{n,m}) \text{ and } Y = (Y_{1,1},\ldots,Y_{1,m},\ldots,Y_{n,1},\ldots,Y_{n,m}).$$

By a standard approximation procedure we may assume that Γ_X and Γ_Y are both positive definite.

For $0 \leqq \theta \leqq 1$, let $\Gamma(\theta) = \theta\Gamma_X + (1-\theta)\Gamma_Y$, and let $\Gamma_X = (r_{\alpha,\beta})$ and $\Gamma_Y = (s_{\alpha,\beta})$ $(1 \leqq \alpha, \beta \leqq nm)$. By assumption (1) of the theorem $r_{\alpha,\alpha} = s_{\alpha,\alpha}$ for all α, therefore

$$\frac{dQ}{d\theta}(Z;\Gamma(\theta)) = \sum_{\alpha<\beta} \frac{\partial Q}{\partial\gamma_{\alpha,\beta}}(Z;\Gamma)\Big|_{\Gamma=\Gamma(\theta)}(r_{\alpha,\beta} - s_{\alpha,\beta}).$$

By assumptions (2) and (3) of the theorem $r_{\alpha,\beta} \leqq s_{\alpha,\beta}$ in case (a), and $r_{\alpha,\beta} \geqq s_{\alpha,\beta}$ in case (b), hence $dQ/d\theta \geqq 0$. Therefore, $Q(Z;\Gamma(1)) \geqq Q(Z;\Gamma(0))$, i.e., $Q(X;\Gamma_X) \geqq Q(Y;\Gamma_Y)$, completing the proof. □

COROLLARY 1.3. *Let $g_{i,j}$ $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$ be increasing functions defined* *on* $(-\infty,\infty)$. *Then, under the assumptions of Theorem* 1.1

$$E\left(\min_{1\leqq i\leqq n}\max_{1\leqq j\leqq m} g_{i,j}(X_{i,j})\right)\geqq E\left(\min_{1\leqq i\leqq n}\max_{1\leqq j\leqq m} g_{i,j}(Y_{i,j})\right).$$

PROOF. Let

$$X=\min_i\max_j g_{i,j}(X_{i,j}),\qquad Y=\min_i\max_j g_{i,j}(Y_{i,j}),$$

and define $g_{i,j}^{-1}(\lambda)$ on $(-\infty,\infty)$ by setting $g_{i,j}^{-1}(\lambda)=\sup\{t;\, g_{i,j}(t)\leqq\lambda\}$. Then,

$$[X\geqq\lambda]=\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}[g_{i,j}(X_{i,j})\geqq\lambda]=\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}[X_{i,j}\geqq g_{i,j}^{-1}(\lambda)],$$

therefore, by Theorem 1.1,

$$\begin{aligned}P(X\geqq\lambda)&=P\left(\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}[X_{i,j}\geqq g_{i,j}^{-1}(\lambda)]\right)\\&\geqq P\left(\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}[Y_{i,j}\geqq g_{i,j}^{-1}(\lambda)]\right)\\&=P(Y\geqq\lambda)\end{aligned}$$

from whence it follows that $E(X)\geqq E(Y)$. □

Theorem 1.4 which follows is an extension of Fernique's theorem [4], and will prove to be the essential ingredient in developing the results of §2.

THEOREM 1.4. *Let* $X_{i,j}$ *and* $Y_{i,j}$ $(1\leqq i\leqq n,\ 1\leqq j\leqq m)$ *be real valued centered Gaussian r.v.s which satisfy the following conditions*:

(i) $E(|X_{i,j}-X_{i,k}|^2)\leqq E(|Y_{i,j}-Y_{i,k}|^2)$ *for all* $1\leqq i\leqq n$, $1\leqq j,\ k\leqq m$,

(ii) $E(|X_{i,j}-X_{l,k}|^2)\geqq E(|Y_{i,j}-Y_{l,k}|^2)$ *for all* $i\neq l$, $1\leqq i,\ l\leqq n$, $1\leqq j,\ k\leqq m$.

Then,

$$E\left(\min_{1\leqq i\leqq n}\max_{1\leqq j\leqq m} X_{i,j}\right)\leqq E\left(\min_{1\leqq i\leqq n}\max_{1\leqq j\leqq m} Y_{i,j}\right).$$

PROOF. We shall continue to use the notation of Theorem 1.1, namely, if $\alpha=(i-1)m+j$ for $1\leqq i\leqq n$, $1\leqq j\leqq m$, the α-th coordinate x_α of a vector x in R^{nm} is also denoted by $x_{i,j}$. Thus, X_α is identified with $X_{i,j}$.

For every $1\leqq\alpha,\ \beta\leqq nm$, let $r_{\alpha,\beta}=E(X_\alpha X_\beta)$, $s_{\alpha,\beta}=E(Y_\alpha Y_\beta)$. For $0\leqq\theta\leqq1$, define $\rho_{\alpha,\beta}(\theta)=\theta r_{\alpha,\beta}+(1-\theta)s_{\alpha,\beta}$. Then the matrix $\Gamma(\theta)=(\rho_{\alpha,\beta}(\theta))_{\alpha,\beta=1}^{nm}$ is the covariance matrix of

$$X(\theta)=(X_{1,1}(\theta),\dots,X_{1,m}(\theta),X_{2,1}(\theta),\dots,X_{2,m}(\theta),\dots,X_{n,1}(\theta),\dots,X_{n,m}(\theta))$$

where $X_{i,j}(\theta)=\theta^{1/2}X_{i,j}+(1-\theta)^{1/2}Y_{i,j}$. We may assume, of course, that the sequence $\{X_{i,j}\}$ is independent of the sequence $\{Y_{i,j}\}$ so that $E(X_{i,j}Y_{l,k})=0$ for all $1\leqq i,\ l\leqq n,\ 1\leqq j,\ k\leqq m$. Also, by standard approximation procedure we may also assume that the matrices $\Gamma(0)$ and $\Gamma(1)$ are both positive definite, and therefore $\Gamma(\theta)$ is positive definite for all θ. Let

$$h(\theta)=E\left(\min_{1\leqq i\leqq n}\max_{1\leqq j\leqq m}X_{i,j}(\theta)\right),$$

and let $g(\theta)$ be the density function of $X(\theta)$. We shall prove that conditions (i) and (ii) of the theorem imply that $h'(\theta)\leqq 0$ for all $0\leqq\theta\leqq 1$; this will show that $h(1)\leqq h(0)$.

We will list the following well-known identities:

$$g_\theta(z)=(2\pi)^{-nm}\int_{R^{nm}}\exp\{i(x,z)-\tfrac{1}{2}(\Gamma(\theta)x,x)\}dx, \tag{1}$$

$$h(\theta)=\int_{R^{nm}}\left(\min_i\max_j x_{i,j}\right)g_\theta(x)dx, \tag{2}$$

$$h'(\theta)=\int_{R^{nm}}\left(\min_i\max_j x_{i,j}\right)\frac{\partial g_\theta(x)}{\partial\theta}dx, \tag{3}$$

$$\frac{\partial g_\theta(x)}{\partial\theta}=-(2\pi)^{-nm}\int_{R^{nm}}\left[\frac{1}{2}\sum_{\alpha,\beta=1}^{nm}y_\alpha y_\beta\frac{d\rho_{\alpha,\beta}(\theta)}{d\theta}\right]\exp\{i(x,y)-\tfrac{1}{2}(\Gamma(\theta)y,y)\}dy, \tag{4}$$

$$\frac{\partial^2 g_\theta(x)}{\partial x_\alpha\partial x_\beta}=-(2\pi)^{-nm}\int_{R^{nm}}y_\alpha y_\beta\exp\{i(x,y)-\tfrac{1}{2}(\Gamma(\theta)y,y)\}dy; \tag{5}$$

therefore

$$\frac{\partial g_\theta(x)}{\partial\theta}=\frac{1}{2}\sum_{\alpha,\beta=1}^{nm}\frac{d\rho_{\alpha,\beta}(\theta)}{d\theta}\cdot\frac{\partial^2 g_\theta(x)}{\partial x_\alpha\partial x_\beta}, \tag{6}$$

$$\begin{aligned}h'(\theta)&=\frac{1}{2}\sum_{\alpha,\beta=1}^{nm}\frac{d\rho_{\alpha,\beta}(\theta)}{d\theta}\int_{R^{nm}}\left(\min_i\max_j x_{i,j}\right)\frac{\partial^2 g_\theta(x)}{\partial x_\alpha\partial x_\beta}dx\\&=\frac{1}{2}\sum_{\alpha,\beta=1}^{nm}(r_{\alpha,\beta}-s_{\alpha,\beta})\int_{R^{nm}}\left(\min_i\max_j x_{i,j}\right)\frac{\partial^2 g_\theta(x)}{\partial x_\alpha\partial x_\beta}dx.\end{aligned} \tag{7}$$

Denote the integral in (7) by $I_{\alpha,\beta}$ (note that we mixed the two notations for the coordinates of x in the integrand of $I_{\alpha,\beta}$). We shall compute $I_{\alpha,\beta}$ for all values of $\alpha,\ \beta,\ 1\leqq\alpha,\ \beta\leqq nm$. It suffices to consider three special cases: (a) $\alpha=\beta=1$,

(b) $\alpha = 1, \beta = 2$, and (c) $\alpha = 1, \beta = m+1$, from which $I_{\alpha,\beta}$ can be determined for all α, β.

We shall first simplify $I_{1,\beta}$ for all β.

Computation of $I_{1,\beta}$: Let $\beta = (i_0 - 1)m + j_0$ where $1 \leqq i_0 \leqq n$, $1 \leqq j_0 \leqq m$. Let us denote

$$\frac{dx}{dx_\alpha} = dx_1 dx_2 \cdots dx_{\alpha-1} dx_{\alpha+1} \cdots dx_{nm},$$

and

$$\frac{dx}{dx_\alpha dx_\beta} = dx_1 \cdots dx_{\alpha-1} dx_{\alpha+1} \cdots dx_{\beta-1} dx_{\beta+1} \cdots dx_{nm}$$

for $\alpha \neq \beta$.

For all $1 \leqq i \leqq n$, $1 \leqq j \leqq m$, let

$$u_i = \min_{\substack{1 \leqq l \leqq n \\ l \neq i}} \max_{1 \leqq j \leqq m} x_{l,j} \quad \text{and} \quad u_{i,j} = \max_{\substack{1 \leqq k \leqq m \\ k \neq j}} x_{i,k}.$$

Then

$$\min_{1 \leqq i \leqq n} \max_{1 \leqq j \leqq m} x_{i,j} = \min\{\max(x_{1,1}, u_{1,1}), u_1\}.$$

There are six cases to consider:

(a) $u_1 \leqq u_{1,1} \leqq x_{1,1}$, then $u_1 = \min_i \max_j x_{i,j}$;

(b) $u_{1,1} \leqq u_1 \leqq x_{1,1}$, then $u_1 = \min_i \max_j x_{i,j}$;

(c) $u_1 \leqq x_{1,1} \leqq u_{1,1}$, then $u_1 = \min_i \max_j x_{i,j}$;

(d) $x_{1,1} \leqq u_1 \leqq u_{1,1}$, then $u_1 = \min_i \max_j x_{i,j}$;

(e) $u_{1,1} \leqq x_{1,1} \leqq u_1$, then $x_{1,1} = \min_i \max_j x_{i,j}$;

(f) $x_{1,1} \leqq u_{1,1} \leqq u_1$, then $u_{1,1} = \min_i \max_j x_{i,j}$.

Let $\beta = (i_0 - 1)m + j_0$, then integrating over the domains (a)+(c)+(d) we obtain

$$\int_{(a)+(c)+(d)} \left(\min_i \max_j x_{i,j}\right) \frac{\partial^2 g_\theta(x)}{\partial x_{1,1} \partial x_{i_0,j_0}} dx = \int_{u_1 \leqq u_{1,1}} u_1 \left(\int_{-\infty}^{\infty} \frac{\partial^2 g_\theta(x)}{\partial x_{1,1} \partial x_{i_0,j_0}} dx_{1,1}\right) \frac{dx}{dx_{1,1}}$$

$$= 0$$

hence

$$I_{1,\beta}=\int_{(b)+(e)+(f)}$$

$$=\int_{u_{1,1}\leqq u_1}\left(u_1\int_{u_1}^{\infty}\frac{\partial^2 g_\theta(x)}{\partial x_{1,1}\partial x_{i_0,j_0}}dx_{1,1}+\int_{u_{1,1}}^{u_1}x_{1,1}\frac{\partial^2 g_\theta(x)}{\partial x_{1,1}\partial x_{i_0,j_0}}dx_{1,1}\right.$$
$$\left.+u_{1,1}\int_{-\infty}^{u_{1,1}}\frac{\partial^2 g_\theta(x)}{\partial x_{1,1}\partial x_{i_0,j_0}}dx_{1,1}\right)\frac{dx}{dx_{1,1}}$$

$$(8)\qquad =-\int_{u_{1,1}\leqq u_1}u_1\frac{\partial g_\theta(x)}{\partial x_{i_0,j_0}}\bigg|_{x_{1,1}=u_1}\frac{dx}{dx_{1,1}}$$
$$+\int_{u_{1,1}\leqq u_1}\left[u_1\frac{\partial g_\theta(x)}{\partial x_{i_0,j_0}}\bigg|_{x_{1,1}=u_1}-u_{1,1}\frac{\partial g_\theta(x)}{\partial x_{i_0,j_0}}\bigg|_{x_{1,1}=u_{1,1}}-\int_{u_{1,1}}^{u_1}\frac{\partial g_\theta}{\partial x_{i_0,j_0}}dx_{1,1}\right]\frac{dx}{dx_{1,1}}$$
$$+\int_{u_{1,1}\leqq u_1}u_{1,1}\frac{\partial g_\theta(x)}{\partial x_{i_0,j_0}}\bigg|_{x_{1,1}=u_{1,1}}\frac{dx}{dx_{1,1}}$$
$$=-\int_{u_{1,1}\leqq u_1}\left(\int_{u_{1,1}}^{u_1}\frac{\partial g_\theta(x)}{\partial x_{i_0,j_0}}dx_{1,1}\right)\frac{dx}{dx_{1,1}}.$$

Computation of $I_{1,1}$: Take in (8) $\beta=1$, then $i_0=j_0=1$, so

$$I_{1,1}=-\int_{u_{1,1}\leqq u_1}g_\theta(x)\big|_{x_{1,1}=u_1}\frac{dx}{dx_{1,1}}+\int_{u_{1,1}\leqq u_1}g_\theta(x)\big|_{x_{1,1}=u_{1,1}}\frac{dx}{dx_{1,1}}$$
$$=B-A$$

where

$$A=\int_{u_{1,1}\leqq u_1}g_\theta(x)\big|_{x_{1,1}=u_1}\frac{dx}{dx_{1,1}},\quad B=\int_{u_{1,1}\leqq u_1}g_\theta(x)\big|_{x_{1,1}=u_{1,1}}\frac{dx}{dx_{1,1}}.$$

To compute B, define for each k, $2\leqq k\leqq m$,

$$B_k=\{x\in R^{nm};\,x_{1,k}\geqq u_{1,k}^{(1)}\},\qquad \text{where } u_{j,k}^{(i)}=\max_{\substack{1\leqq l\leqq m\\ l\neq j,k}}x_{i,l},$$

then on B_k, $u_{1,1}=x_{1,k}$, and the condition $u_{1,1}\leqq u_1$ implies on B_k, $u_{1,k}^{(1)}\leqq x_{1,k}\leqq u_1$, hence

$$B=\sum_{k=2}^{m}\int_{u_{1,k}^{(1)}\leqq u_1}\left(\int_{u_{1,k}^{(1)}}^{u_1}g_\theta(x)\big|_{x_{1,1}=x_{1,k}=\xi}d\xi\right)\frac{dx}{dx_{1,1}dx_{1,k}}.$$

Now we compute A. For all $1\leqq i,\ l\leqq n$, $i\neq l$, let

$$u^{i,l}=\min_{\substack{1\leqq r\leqq n\\ r\neq i,l}}\max_k x_{r,k}.$$

Let

$$C_l = \left\{ x \in R^{nm} ; \max_j x_{l,j} \leqq \min_{\substack{2 \leqq r \leqq n \\ r \neq l}} \max_j x_{r,j} \overset{\text{def}}{=} u^{1,l} \right\}, \qquad l = 2, 3, \dots, n.$$

Then $R^{nm} = \bigcup_{l=2}^{n} C_l$ and distinct C_l's have disjoint interiors. Moreover, each C_l is also the union of sets which have pairwise disjoint interiors, namely, $C_l = \bigcup_{k=1}^{m} B_{l,k}$ where

$$B_{l,k} = C_l \cap \left\{ x \in R^{nm} ; x_{l,k} \geqq \max_{\substack{1 \leqq j \leqq m \\ j \neq k}} x_{l,j} \right\}.$$

Since $R^{nm} = \bigcup_{l=2}^{n} \bigcup_{k=1}^{m} B_{l,k}$ we obtain

$$A = \sum_{l=2}^{n} \sum_{k=1}^{m} \int_{B_{l,k} \cap \{u_{1,1} \leqq u_1\}} g_\theta(x) \big|_{x_{1,1} = u_1} \frac{dx}{dx_{1,1}}.$$

But on $B_{l,k} \cap \{u_{1,1} \leqq u_1\}$ we have $u_1 = x_{l,k}$, hence $x_{l,k} \geqq u_{1,1}$, also

$$x_{l,k} \geqq \max_{\substack{1 \leqq j \leqq m \\ j \neq k}} x_{l,j} \overset{\text{def}}{=} u_{l,k} \quad \text{and} \quad x_{l,k} \leqq \min_{\substack{2 \leqq r \leqq n \\ r \neq l}} \max_j x_{r,j} = u^{1,l}.$$

Thus we have on $B_{l,k} \cap \{u_{1,1} \leqq u_1\}$ that $\max\{u_{1,1}, u_{l,k}\} \leqq x_{l,k} \leqq u^{1,l}$ (and this inequality in fact defines the set $B_{l,k} \cap \{u_{1,1} \leqq u_1\}$), therefore denoting $\max\{a, b\}$ by $a \vee b$ we obtain

$$A = \sum_{l=2}^{n} \sum_{k=1}^{m} \int_{u_{1,1} \vee u_{l,k} \leqq u^{1,l}} \left(\int_{u_{1,1} \vee u_{l,k}}^{u^{1,l}} g_\theta(x) \big|_{x_{1,1} = x_{l,k} = \xi} d\xi \right) \frac{dx}{dx_{1,1} dx_{l,k}},$$

and from these identities we obtain

$$\begin{aligned} I_{1,1} = & \sum_{k=2}^{m} \int_{u_{1,k}^{(1)} \leqq u_1} \left(\int_{u_{1,k}^{(1)}}^{u_1} g_\theta(x) \big|_{x_{1,1} = x_{1,k} = \xi} d\xi \right) \frac{dx}{dx_{1,1} dx_{1,k}} \\ & - \sum_{l=2}^{m} \sum_{k=1}^{m} \int_{u_{1,1} \vee u_{l,k} \leqq u^{1,l}} \left(\int_{u_{1,1} \vee u_{l,k}}^{u^{1,l}} g_\theta(x) \big|_{x_{1,1} = x_{l,k} = \xi} d\xi \right) \frac{dx}{dx_{1,1} dx_{l,k}}. \end{aligned}$$

In the same way we can determine $I_{\alpha,\alpha}$ for each $1 \leqq \alpha \leqq nm$; setting $\alpha = (i-1)m + j$ ($\leqq i \leqq n$, $1 \leqq j \leqq m$), we obtain

$$\begin{aligned} I_{\alpha,\alpha} = & \sum_{\substack{k=1 \\ k \neq j}}^{m} \int_{u_{j,k}^{(i)} \leqq u_i} \left(\int_{u_{j,k}^{(i)}}^{u_i} g_\theta(x) \big|_{x_{i,j} = x_{i,k} = \xi} d\xi \right) \frac{dx}{dx_{i,j} dx_{i,k}} \\ & - \sum_{\substack{l=1 \\ l \neq i}}^{n} \sum_{k=1}^{m} \int_{u_{i,j} \vee u_{l,k} \leqq u^{i,l}} \left(\int_{u_{i,j} \vee u_{l,k}}^{u^{i,l}} g_\theta(x) \big|_{x_{i,j} = x_{l,k} = \xi} d\xi \right) \frac{dx}{dx_{i,j} dx_{l,k}}. \end{aligned} \tag{9}$$

Computation of $I_{1,2}$: If we take in (8) $i_0 = 1$, $j_0 = 2$ we get that for $\alpha = 1$, $\beta = 2$

$$I_{1,2} = -\int_{u_{1,1} \leqq u_1} \left(\int_{u_{1,1}}^{u_1} \frac{\partial g_\theta(x)}{\partial x_{1,2}} dx_{1,1} \right) dx_{1,2} \frac{dx}{dx_{1,1} dx_{1,2}}.$$

Now,

$$u_{1,1} = \max_{2 \leqq j \leqq m} x_{1,j} = x_{1,2} \vee u_{1,2}^{(1)}.$$

There are two cases to consider here:

(1) $u_{1,2}^{(1)} \leqq x_{1,2}$. Then $u_{1,1} \leqq u_1$ implies $x_{1,2} \leqq u_1$, and the condition $u_{1,1} \leqq x_{1,1} \leqq u_1$ implies $x_{1,2} \leqq x_{1,1} \leqq u_1$. That is: $u_{1,2}^{(1)} \leqq x_{1,2} \leqq x_{1,1} \leqq u_1$.

(2) $u_{1,2}^{(1)} \geqq x_{1,2}$. Then $u_{1,1} \leqq u_1$ implies $u_{1,2}^{(1)} \leqq u_1$, and the condition $u_{1,1} \leqq x_{1,1} \leqq u_1$ implies $u_{1,2}^{(1)} \leqq x_{1,1} \leqq u_1$. That is: $x_{1,2} \leqq u_{1,2}^{(1)} \leqq x_{1,1} \leqq u_1$.

Therefore, changing the order of integration in $I_{1,2}$ we obtain

$$\begin{aligned} I_{1,2} = &-\int_{u_{1,2}^{(1)} \leqq u_1} \left[\int_{u_{1,2}^{(1)}}^{u_1} \int_{u_{1,2}^{(1)}}^{x_{1,1}} \frac{\partial g_\theta(x)}{\partial x_{1,2}} dx_{1,2} dx_{1,1} \right. \\ &\left. + \int_{u_{1,2}^{(1)}}^{u_1} \int_{-\infty}^{u_{1,2}^{(1)}} \frac{\partial g_\theta(x)}{\partial x_{1,2}} dx_{1,2} dx_{1,1} \right] \frac{dx}{dx_{1,1} dx_{1,2}} \\ = &-\int_{u_{1,2}^{(1)} \leqq u_1} \left(\int_{u_{1,2}^{(1)}}^{u_1} g_\theta(x)\big|_{x_{1,1} = x_{1,2} = \xi} d\xi \right) \frac{dx}{dx_{1,1} dx_{1,2}}. \end{aligned}$$

This implies that for any $1 \leqq \alpha, \beta \leqq nm$, if $\alpha = (i-1)m + j$ and $\beta = (i-1)m + k$, where $1 \leqq i \leqq n$ and $1 \leqq j \neq k \leqq m$, then

$$I_{\alpha,\beta} = -\int_{u_{j,k}^{(i)} \leqq u_i} \left(\int_{u_{j,k}^{(i)}}^{u_i} g_\theta(x)\big|_{x_{i,j} = x_{i,k} = \xi} d\xi \right) \frac{dx}{dx_{i,j} dx_{i,k}}. \tag{10}$$

Computation of $I_{1,m+1}$: By equation (8)

$$I_{1,m+1} = -\int_{u_{1,1} \leqq u_1} \left(\int_{u_{1,1}}^{u_1} \frac{\partial g_\theta(x)}{\partial x_{2,1}} dx_{1,1} \right) \frac{dx}{dx_{1,1}}.$$

Recall that $u_1 = \min\{u^{1,2}, \max\{x_{2,1}, u_{2,1}\}\}$, therefore there are six cases to consider here:

(a) $u^{1,2} \leqq u_{2,1} \leqq x_{2,1}$, here $u_1 = u^{1,2}$;

(b) $u^{1,2} \leqq x_{2,1} \leqq u_{2,1}$, here $u_1 = u^{1,2}$;

(c) $x_{2,1} \leqq u^{1,2} \leqq u_{2,1}$, here $u_1 = u^{1,2}$;

(d) $u_{2,1} \leqq u^{1,2} \leqq x_{2,1}$, here $u_1 = u^{1,2}$;

(e) $u_{2,1} \leqq x_{2,1} \leqq u^{1,2}$, here $u_1 = x_{2,1}$;

(f) $x_{2,1} \leqq u_{2,1} \leqq u^{1,2}$, here $u_1 = u_{2,1}$.

We obtain therefore

$$I_{1,m+1} = -\int_{u_{1,1}\leqq u^{1,2}\leqq u_{2,1}} \left(\int_{u_{2,1}}^{\infty}\int_{u_{1,1}}^{u^{1,2}} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{1,1}dx_{2,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\leqq u^{1,2}\leqq u_{2,1}} \left(\int_{u^{1,2}}^{u_{2,1}}\int_{u_{1,1}}^{u^{1,2}} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{1,1}dx_{2,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\leqq u^{1,2}\leqq u_{2,1}} \left(\int_{-\infty}^{u^{1,2}}\int_{u_{1,1}}^{u^{1,2}} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{1,1}dx_{2,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\vee u_{2,1}\leqq u^{1,2}} \left(\int_{u^{1,2}}^{\infty}\int_{u_{1,1}}^{u^{1,2}} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{1,1}dx_{2,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\vee u_{2,1}\leqq u^{1,2}} \left(\int_{u_{1,1}\vee u_{2,1}}^{u^{1,2}}\int_{u_{1,1}}^{x_{2,1}} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{1,1}dx_{2,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\leqq u_{2,1}\leqq u^{1,2}} \left(\int_{-\infty}^{u_{2,1}}\int_{u_{1,1}}^{u_{2,1}} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{1,1}dx_{2,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}.$$

Changing the order of integration inside the first three integrals, we see that their sum is zero since

$$\int_{-\infty}^{\infty} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{2,1} = 0.$$

Hence

$$I_{1,m+1} = \int_{u_{1,1}\vee u_{2,1}\leqq u^{1,2}} \left(\int_{u_{1,1}}^{u^{1,2}} g_\theta(x)\big|_{x_{2,1}=u^{1,2}}\,dx_{1,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\vee u_{2,1}\leqq u^{1,2}} \left[\int_{u_{1,1}}^{u_{1,1}\vee u_{2,1}}\int_{u_{1,1}\vee u_{2,1}}^{u^{1,2}} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{2,1}dx_{1,1}\right.$$

$$\left.+\int_{u_{1,1}\vee u_{2,1}}^{u^{1,2}}\int_{x_{1,1}}^{u^{1,2}} \frac{\partial g_\theta(x)}{\partial x_{2,1}}\,dx_{2,1}dx_{1,1}\right]\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\leqq u_{2,1}\leqq u^{1,2}} \left(\int_{u_{1,1}}^{u_{2,1}} g_\theta(x)\big|_{x_{2,1}=u_{2,1}}\,dx_{1,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}.$$

We have to consider the cases $u_{2,1}\leqq u_{1,1}$ and $u_{1,1}\leqq u_{2,1}$. These cases lead to the following computation:

$$I_{1,m+1} = \int_{u_{1,1}\leqq u_{2,1}\leqq u^{1,2}} \left(\int_{u_{1,1}}^{u^{1,2}} g_\theta(x)\big|_{x_{2,1}=u^{1,2}}\,dx_{1,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$+\int_{u_{2,1}\leqq u_{1,1}\leqq u^{1,2}} \left(\int_{u_{1,1}}^{u^{1,2}} g_\theta(x)\big|_{x_{2,1}=u^{1,2}}\,dx_{1,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\leqq u_{2,1}\leqq u^{1,2}}\Bigg\{\int_{u_{1,1}}^{u_{2,1}}[g_\theta(x)|_{x_{2,1}=u^{1,2}}-g_\theta(x)|_{x_{2,1}=u_{2,1}}]dx_{1,1}$$

$$+\int_{u_{2,1}}^{u^{1,2}}[g_\theta(x)|_{x_{2,1}=u^{1,2}}-g_\theta(x)|_{x_{2,1}=x_{1,1}}]dx_{1,1}\Bigg\}\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{2,1}\leqq u_{1,1}\leqq u^{1,2}}\Bigg\{\int_{u_{1,1}}^{u_{1,1}}\int_{u_{1,1}}^{u^{1,2}}\frac{\partial g_\theta(x)}{\partial x_{2,1}}dx_{2,1}dx_{1,1}$$

$$+\int_{u_{1,1}}^{u^{1,2}}[g_\theta(x)|_{x_{2,1}=u^{1,2}}-g_\theta(x)|_{x_{2,1}=x_{1,1}}]dx_{1,1}\Bigg\}\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$-\int_{u_{1,1}\leqq u_{2,1}\leqq u^{1,2}}\left(\int_{u_{1,1}}^{u_{2,1}}g_\theta(x)|_{x_{2,1}=u_{2,1}}dx_{1,1}\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$=\int_{u_{1,1}\leqq u_{2,1}\leqq u^{1,2}}\left(\int_{u_{2,1}}^{u^{1,2}}g_\theta(x)|_{x_{1,1}=x_{2,1}=\xi}d\xi\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$+\int_{u_{2,1}\leqq u_{1,1}\leqq u^{1,2}}\left(\int_{u_{1,1}}^{u^{1,2}}g_\theta(x)|_{x_{1,1}=x_{2,1}=\xi}d\xi\right)\frac{dx}{dx_{1,1}dx_{2,1}}$$

$$=\int_{u_{1,1}\vee u_{2,1}\leqq u^{1,2}}\left(\int_{u_{1,1}\vee u_{2,1}}^{u^{1,2}}g_\theta(x)|_{x_{1,1}=x_{2,1}=\xi}d\xi\right)\frac{dx}{dx_{1,1}dx_{2,1}}.$$

From this we obtain for every $1\leqq\alpha,\ \beta\leqq nm$, if $\alpha=(i-1)m+j$ and $\beta=(l-1)m+k$, where $1\leqq i\neq l\leqq n$, $1\leqq j,\ k\leqq m$, that

$$I_{\alpha,\beta}=\int_{u_{i,j}\vee u_{l,k}\leqq u^{i,l}}\left(\int_{u_{i,j}\vee u_{l,k}}^{u^{i,l}}g_\theta(x)|_{x_{i,j}=x_{l,k}=\xi}d\xi\right)\frac{dx}{dx_{i,j}dx_{l,k}}. \tag{11}$$

From equation (7)

$$\frac{dh}{d\theta}=\frac{1}{2}\sum_{\alpha=1}^{nm}(r_{\alpha,\alpha}-s_{\alpha,\alpha})I_{\alpha,\alpha}$$

$$+\frac{1}{2}\sum_{i=1}^{m}\sum_{\substack{1\leqq j,k\leqq m\\ j\neq k}}(r_{(i-1)m+j,(i-1)m+k}-s_{(i-1)m+j,(i-1)m+k})I_{(i-1)m+j,(i-1)m+k}$$

$$+\frac{1}{2}\sum_{\substack{1\leqq i,l\leqq n\\ i\neq l}}\sum_{j,k=1}^{m}(r_{(i-1)m+j,(l-1)m+k}-s_{(i-1)m+j,(l-1)m+k})I_{(i-1)m+j,(l-1)m+k}$$

$$=(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})$$

where

$$(\mathrm{I})=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}(r_{(i-1)m+j,(i-1)m+j}-s_{(i-1)m+j,(i-1)m+j})\sum_{\substack{k=1\\ k\neq j}}^{m}\int_{u_{j,k}^{(i)}\leqq u_i}\left(\int_{u_{j,k}^{(i)}}^{u_i}g_\theta\,|_{x_{i,j}=x_{i,k}=\xi}d\xi\right)\frac{dx}{dx_{i,j}dx_{i,k}},$$

$$(\mathrm{II}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}(r_{(i-1)m+j,(i-1)m+j} - s_{(i-1)m+j,(i-1)m+j})$$

$$\times \sum_{\substack{l=1 \\ l\neq i}}^{n}\sum_{k=1}^{m}\int_{u_{l,k}\vee u_{i,j}\leqq u^{i,l}}\left(\int_{u_{l,k}\vee u_{i,j}}^{u^{i,l}} g_\theta\,\Big|_{x_{i,j}=x_{l,k}=\xi}\,d\xi\right)\frac{dx}{dx_{i,j}dx_{l,k}},$$

$$(\mathrm{III}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{1\leqq j,k\leqq m \\ j\neq k}}(r_{(i-1)m+j,(i-1)m+k} - s_{(i-1)m+j,(i-1)m+k})$$

$$\times \int_{u_{j,k}^{(i)}\leqq u_i}\left(\int_{u_{j,k}^{(i)}}^{u_i} g_\theta\,\Big|_{x_{i,j}=x_{i,k}=\xi}\,d\xi\right)\frac{dx}{dx_{i,j}dx_{i,k}},$$

$$(\mathrm{IV}) = \frac{1}{2}\sum_{\substack{1\leqq i,l\leqq n \\ i\neq l}}\sum_{j,k=1}^{m}(r_{(i-1)m+j,(l-1)m+k} - s_{(i-1)m+j,(l-1)m+k})$$

$$\times \int_{u_{i,j}\vee u_{l,k}\leqq u^{i,l}}\left(\int_{u_{i,j}\vee u_{l,k}}^{u^{i,l}} g_\theta\,\Big|_{x_{i,j}=x_{l,k}=\xi}\,d\xi\right)\frac{dx}{dx_{i,j}dx_{l,k}}.$$

But we can write (I) as

$$(\mathrm{I}) = \frac{1}{4}\sum_{i=1}^{n}\sum_{\substack{1\leqq j,k\leqq m \\ j\neq k}}(r_{(i-1)m+j,(i-1)m+j} + r_{(i-1)m+k,(i-1)m+k} - s_{(i-1)m+j,(i-1)m+j}$$

$$- s_{(i-1)m+k,(i-1)m+k})\int_{u_{j,k}^{(i)}\leqq u_i}\left(\int_{u_{j,k}^{(i)}}^{u_i} g_\theta\,\Big|_{x_{i,j}=x_{i,k}=\xi}\,d\xi\right)\frac{dx}{dx_{i,j}dx_{i,k}}$$

and since the respective integrals that appear in the sums (I) and (III) are the same and non-negative, because $g_\theta \geqq 0$, it follows that (I) + (III) is $\leqq 0$ if the coefficients are all $\leqq 0$, that is, if for all $1\leqq i\leqq n$, and $1\leqq j\neq k\leqq m$ we have

$$0 \geqq r_{(i-1)m+j,(i-1)m+j} + r_{(i-1)m+k,(i-1)m+k} - s_{(i-1)m+j,(i-1)m+j} - s_{(i-1)m+k,(i-1)m+k}$$
$$- 2r_{(i-1)m+j,(i-1)m+k} + 2s_{(i-1)m+j,(i-1)m+k},$$

which is precisely inequality (i) of the assumption of the theorem.

Similarly, we can write (II) as

$$(\mathrm{II}) = -\frac{1}{4}\sum_{\substack{1\leqq i,l\leqq n \\ i\neq l}}\sum_{j,k=1}^{m}(r_{(i-1)m+j,(i-1)m+j} + r_{(l-1)m+k,(l-1)m+k} - s_{(i-1)m+j,(i-1)m+j}$$

$$- s_{(l-1)m+k,(l-1)m+k})\int_{u_{l,k}\vee u_{ij}\leqq u^{i,l}}\left(\int_{u_{l,k}\vee u_{i,j}}^{u^{i,l}} g_\theta\,\Big|_{x_{i,j}=x_{l,k}=\xi}\,d\xi\right)\frac{dx}{dx_{i,j}dx_{l,k}}.$$

(II) therefore contains the same integrals as (IV), and since $g_\theta \geqq 0$, (II) + (IV) will be non-positive if all the respective coefficients are $\leqq 0$, that is, if for all $1\leqq i\neq l\leqq n$, $1\leqq j,k\leqq m$

$$0 \geqq -r_{(i-1)m+j,(i-1)m+j} - r_{(l-1)m+k,(l-1)m+k} + s_{(i-1)m+j,(i-1)m+j} + s_{(l-1)m+k,(l-1)m+k}$$
$$+ 2r_{(i-1)m+j,(l-1)m+k} - 2s_{(i-1)m+j,(l-1)m+k},$$

which is exactly assumption (ii) of the theorem. Hence $dh/d\theta \leqq 0$ for all θ, implying in particular that $h(0) \geqq h(1)$, that is

$$E\left(\min_i \max_j X_{i,j}\right) \leqq E\left(\min_i \max_j Y_{i,j}\right). \qquad \square$$

REMARK 1.5. Fernique's theorem follows as a corollary of Theorem 1.4, because this is exactly the case when $n = 1$, so (ii) of Theorem 1.4 does not appear, and thus from (i) alone (for $n = 1$) it follows that

$$E\left(\max_j X_{1,j}\right) \leqq E\left(\max_j Y_{1,j}\right).$$

§2. Applications and Dvoretzky's theorem

Let $\{g_{i,j}\}$ $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$, $\{h_i\}_1^n$ and $\{g_i\}_1^m$ always denote independent sets of orthonormal Gaussian r.v.s.

THEOREM 2.1. *Let*

$$\mathscr{E} = \{\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\} \subset R^n \quad \textit{and} \quad \Theta = \{\theta = (\theta_1, \theta_2, \ldots, \theta_m)\} \subset R^m$$

be compact subsets of points, and let

$$X_{\varepsilon,\theta} = \|\varepsilon\|_2 \sum_{j=1}^m \theta_j g_j + \theta_0 \sum_{i=1}^n \varepsilon_i h_i \qquad (\textit{where } \theta_0 = \max\{\|\theta\|_2; \theta \in \Theta\})$$

and

$$Y_{\varepsilon,\theta} = \sum_{i=1}^n \sum_{j=1}^m \varepsilon_i \theta_j g_{i,j}.$$

Then

$$E\left(\min_{\varepsilon\in\mathscr{E}} \max_{\theta\in\Theta} X_{\varepsilon,\theta}\right) \leqq E\left(\min_{\varepsilon\in\mathscr{E}} \max_{\theta\in\Theta} Y_{\varepsilon,\theta}\right) \leqq E\left(\max_{\varepsilon\in\mathscr{E}} \max_{\theta\in\Theta} Y_{\varepsilon,\theta}\right) \leqq E\left(\max_{\varepsilon\in\mathscr{E}} \max_{\theta\in\Theta} X_{\varepsilon,\theta}\right).$$

PROOF. For arbitrary ε, $\tilde\varepsilon$ in $\mathscr{E}$, and θ, $\tilde\theta$ in Θ we have that

$$E(|Y_{\varepsilon,\theta} - Y_{\tilde\varepsilon,\tilde\theta}|^2) = \sum_{i=1}^n \sum_{j=1}^m (\varepsilon_i\theta_j - \tilde\varepsilon_i\tilde\theta_j)^2$$
$$= \|\varepsilon\|_2^2\|\theta\|_2^2 + \|\tilde\varepsilon\|_2^2\|\tilde\theta\|_2^2 - 2(\varepsilon, \tilde\varepsilon)(\theta, \tilde\theta)$$

and

$$E(|X_{\varepsilon,\theta}-X_{\tilde\varepsilon,\tilde\theta}|^2)=\sum_{j=1}^{m}(\|\varepsilon\|_2\theta_j-\|\tilde\varepsilon\|_2\tilde\theta_j)^2+\theta_0^2\sum_{i=1}^{n}(\varepsilon_i-\tilde\varepsilon_i)^2$$

$$=\|\varepsilon\|_2^2\|\theta\|_2^2+\|\tilde\varepsilon\|_2^2\|\tilde\theta\|_2^2-2\|\varepsilon\|_2\|\tilde\varepsilon\|_2(\theta,\tilde\theta)$$

$$+\theta_0^2[\|\varepsilon\|_2^2+\|\tilde\varepsilon\|_2^2-2(\varepsilon,\tilde\varepsilon)],$$

hence

$$E(|X_{\varepsilon,\theta}-X_{\tilde\varepsilon,\tilde\theta}|^2)\geqq E(|Y_{\varepsilon,\theta}-Y_{\tilde\varepsilon,\tilde\theta}|^2) \tag{2.1}$$

if and only if

$$\theta_0^2[\|\varepsilon\|_2^2+\|\tilde\varepsilon\|_2^2-2(\varepsilon,\tilde\varepsilon)]-2(\theta,\tilde\theta)[\|\varepsilon\|_2\|\tilde\varepsilon\|_2-(\varepsilon,\tilde\varepsilon)]\geqq 0. \tag{2.2}$$

But since $\|\varepsilon\|_2\|\tilde\varepsilon\|_2-(\varepsilon,\tilde\varepsilon)\geqq 0$ the l.h.s. of (2.2) is minimized when $(\theta,\tilde\theta)=\theta_0^2$, and then (2.2) is reduced to the inequality

$$\theta_0^2[\|\varepsilon\|_2^2+\|\tilde\varepsilon\|_2^2-2\|\varepsilon\|_2\|\tilde\varepsilon\|_2]\geqq 0$$

which is obviously true for all ε, $\tilde\varepsilon$ in $\mathscr{E}$.

By Fernique's theorem, or Remark 1.5, it follows that

$$E\left(\max_{\varepsilon\in\mathscr{E}}\max_{\theta\in\Theta}X_{\varepsilon,\theta}\right)\geqq E\left(\max_{\varepsilon\in\mathscr{E}}\max_{\theta\in\Theta}Y_{\varepsilon,\theta}\right).$$

It remains to show that

$$E\left(\min_{\varepsilon\in\mathscr{E}}\max_{\theta\in\Theta}X_{\varepsilon,\theta}\right)\leqq E\left(\min_{\varepsilon\in\mathscr{E}}\max_{\theta\in\Theta}Y_{\varepsilon,\theta}\right);$$

this will follow from Theorem 1.4 if, in addition to inequality (2.1), we shall also show that

$$E(|X_{\varepsilon,\theta}-X_{\varepsilon,\tilde\theta}|^2)\leqq E(|Y_{\varepsilon,\theta}-Y_{\varepsilon,\tilde\theta}|^2)\qquad\text{for all }\varepsilon\in\mathscr{E}\text{ and }\theta,\tilde\theta\in\Theta.$$

But, in fact, we have that

$$E(|X_{\varepsilon,\theta}-X_{\varepsilon,\tilde\theta}|^2)=\|\varepsilon\|_2^2\sum_{j=1}^{m}(\theta_j-\tilde\theta_j)^2=E(|Y_{\varepsilon,\theta}-Y_{\varepsilon,\tilde\theta}|^2)$$

and this completes the proof. □

We now illustrate the preceding theorem by an example:

EXAMPLE. Let $\mathscr{E}$ consist of all 2^n vectors of the form $(\pm 1,\dots,\pm 1)$ in R^n, and θ consist of all 2^m vectors of the form $(\pm 1,\dots,\pm 1)$ in R^m. Let $X_{\varepsilon,\theta}$ and $Y_{\varepsilon,\theta}$ be as in Theorem 2.1. Then $\theta_0=\sqrt{m}$ and $\|\varepsilon\|_2=\sqrt{n}$ for all $\varepsilon\in\mathscr{E}$, so

$$E\left(\max_{\varepsilon}\max_{\theta} X_{\varepsilon,\theta}\right) = E\left(\sqrt{n}\sum_{j=1}^{m}|g_j| + \sqrt{m}\sum_{i=1}^{n}|h_i|\right)$$

$$= \sqrt{\frac{2}{\pi}}(m\sqrt{n} + n\sqrt{m}) = \sqrt{\frac{2n}{\pi}}\, m\left(1+\sqrt{\frac{n}{m}}\right)$$

and

$$E\left(\min_{\varepsilon}\max_{\theta} X_{\varepsilon,\theta}\right) = E\left(\sqrt{n}\sum_{j=1}^{m}|g_j| - \sqrt{m}\sum_{i=1}^{n}|h_i|\right) = \sqrt{\frac{2n}{\pi}}\, m\left(1-\sqrt{\frac{n}{m}}\right),$$

therefore, by Theorem 2.1

$$E\left(\max_{\varepsilon}\sum_{j=1}^{m}\left|\sum_{i=1}^{n}\varepsilon_i g_{i,j}\right|\right) \leqq \sqrt{\frac{2n}{\pi}}\, m\left(1+\sqrt{\frac{n}{m}}\right)$$

and

$$E\left(\min_{\varepsilon}\sum_{j=1}^{m}\left|\sum_{i=1}^{n}\varepsilon_i g_{i,j}\right|\right) \geqq \sqrt{\frac{2n}{\pi}}\, m\left(1-\sqrt{\frac{n}{m}}\right).$$

Of course, it is also obvious that for all $\varepsilon \in \mathscr{E}$

$$E\left(\min_{\varepsilon}\sum_{j=1}^{m}\left|\sum_{i=1}^{n}\varepsilon_i g_{i,j}\right|\right) \leqq E\left(\sum_{j=1}^{m}\left|\sum_{i=1}^{n}\varepsilon_i g_{i,j}\right|\right) = \sqrt{\frac{2n}{\pi}}\, m.$$

Using Corollary 1.3 and slightly modifying $X_{\varepsilon,\theta}$ of Theorem 2.1, we obtain a general useful inequality.

THEOREM 2.3. *Let $\mathscr{E} \subset R^n$, $\Theta \subset R^m$ be finite sets and*

$$X_{\varepsilon,\theta} = \|\varepsilon\|_2 \sum_{j=1}^{m}\theta_j g_j + \|\theta\|_2\sum_{i=1}^{n}\varepsilon_i h_i, \quad Y_{\varepsilon,\theta} = \sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_i\theta_j g_{i,j},$$

$$Z_{\varepsilon,\theta} = Y_{\varepsilon,\theta} + \|\varepsilon\|_2\|\theta\|_2 g \qquad (\varepsilon \in \mathscr{E},\ \theta \in \Theta),$$

where g is a standard normalized Gaussian variable independent of the $g_{i,j}$. If $f_{\varepsilon,\theta}(t)$ ($\varepsilon \in \mathscr{E}$, $\theta \in \Theta$, $-\infty < t < \infty$) are increasing functions, then

$$E\left(\min_{\varepsilon}\max_{\theta} f_{\varepsilon,\theta}(X_{\varepsilon,\theta})\right) \leqq E\left(\min_{\varepsilon}\max_{\theta} f_{\varepsilon,\theta}(Z_{\varepsilon,\theta})\right)$$

$$\leqq E\left(\max_{\varepsilon,\theta} f_{\varepsilon,\theta}(Z_{\varepsilon,\theta})\right)$$

$$\leqq E\left(\max_{\varepsilon,\theta} f_{\varepsilon,\theta}(X_{\varepsilon,\theta})\right).$$

PROOF. We shall first verify the following inequalities:

(I) $E(X_{\varepsilon,\theta}X_{\varepsilon,\tilde\theta}) = E(Z_{\varepsilon,\theta}Z_{\varepsilon,\tilde\theta})$ for all $\varepsilon \in \mathscr{E}$, $\theta, \tilde\theta \in \Theta$,

(II) $E(X_{\varepsilon,\theta}X_{\tilde\varepsilon,\tilde\theta}) \leqq E(Z_{\varepsilon,\theta}Z_{\tilde\varepsilon,\tilde\theta})$ for all $\varepsilon, \tilde\varepsilon \in \mathscr{E}$, $\theta, \tilde\theta \in \Theta$.

Indeed, both sides of (I) are equal to $\|\varepsilon\|_2^2(\theta, \tilde\theta) + \|\varepsilon\|_2^2\|\theta\|_2\|\tilde\theta\|_2$, and it is easy to see that

$$E(Z_{\varepsilon,\theta}Z_{\tilde\varepsilon,\tilde\theta}) - E(X_{\varepsilon,\theta}X_{\tilde\varepsilon,\tilde\theta}) = \|\varepsilon\|_2\|\tilde\varepsilon\|_2\|\theta\|_2\|\tilde\theta\|_2 - \|\theta\|_2\|\tilde\theta\|_2(\varepsilon, \tilde\varepsilon)$$
$$- (\theta, \tilde\theta)[\|\varepsilon\|_2\|\tilde\varepsilon\|_2 - (\varepsilon, \tilde\varepsilon)]$$

but since $(\theta, \tilde\theta) \leqq \|\theta\|_2\|\tilde\theta\|_2$ and $\|\varepsilon\|_2\|\tilde\varepsilon\|_2 \geqq (\varepsilon, \tilde\varepsilon)$, the above expression is greater than or equal to

$$\|\varepsilon\|_2\|\theta\|_2\|\tilde\varepsilon\|_2\|\tilde\theta\|_2 - \|\theta\|_2\|\tilde\theta\|_2(\varepsilon, \tilde\varepsilon) - \|\theta\|_2\|\tilde\theta\|_2[\|\varepsilon\|_2\|\tilde\varepsilon\|_2 - (\varepsilon, \tilde\varepsilon)] \equiv 0.$$

The validity of inequalities (I) and (II) implies that inequalities (1)–(3) of Theorem 1.1 are valid as well, where $X_{\varepsilon,\theta}$ replaces $Y_{i,j}$ and $Z_{\varepsilon,\theta}$ replaces $X_{i,j}$, hence by Corollary 1.3 it follows that

$$E\left(\min_{\varepsilon}\max_{\theta} f_{\varepsilon,\theta}(Z_{\varepsilon,\theta})\right) \geqq E\left(\min_{\varepsilon}\max_{\theta} f_{\varepsilon,\theta}(X_{\varepsilon,\theta})\right).$$

The fact that $E(X^2_{\varepsilon,\theta}) = E(Y^2_{\varepsilon,\theta})$ and $E(X_{\varepsilon,\theta}X_{\tilde\varepsilon,\tilde\theta}) \leqq E(Z_{\varepsilon,\theta}Z_{\tilde\varepsilon,\tilde\theta})$ for all $\varepsilon, \tilde\varepsilon \in \mathscr{E}$, $\theta, \tilde\theta \in \Theta$, implies by Slepian's lemma, which is a consequence of Theorem 1.1, that

$$P\left(\bigcup_{\varepsilon,\theta}[X_{\varepsilon,\theta} \geqq \lambda_{\varepsilon,\theta}]\right) \geqq P\left(\bigcup_{\varepsilon,\theta}[Z_{\varepsilon,\theta} \geqq \lambda_{\varepsilon,\theta}]\right)$$

for all real scalars $\lambda_{\varepsilon,\theta}$, and as in Corollary 1.3 this proves that

$$E\left(\max_{\varepsilon,\theta} f_{\varepsilon,\theta}(X_{\varepsilon,\theta})\right) \geqq E\left(\max_{\varepsilon,\theta} f_{\varepsilon,\theta}(Z_{\varepsilon,\theta})\right). \qquad \square$$

REMARK 1. If we replace $\|\theta\|_2$ by θ_0 in the definitions of $X_{\varepsilon,\theta}$ and $Z_{\varepsilon,\theta}$, then the inequality of Theorem 2.3 remains the same.

REMARK 2. We note also that by varying the sets $\mathscr{E}$ and Θ in Theorems 2.1 and 2.2, many other interesting inequalities can be formed. One may take, for example,

$$\mathscr{E} = \{\varepsilon \in R^n; \|\varepsilon\|_2 = 1\} \quad \text{and} \quad \Theta = \{(y^*(y_1), \ldots, y^*(y_m)); y^* \in Y^*, \|y^*\| = 1\}$$

where $\{y_i\}_1^m$ is a fixed set of vectors in a Banach space Y; we'll show that by using Theorem 2.1 this leads to a proof of Dvoretzky's theorem.

The right-hand side of the inequality in Theorem 2.1 refines an inequality of Chevet [1], in that it replaces the constant $\sqrt{2}$ by 1, namely:

COROLLARY 2.4. *Let* $\{x_i\}_{i=1}^n \subset X$ *and* $\{y_j\}_{j=1}^m \subset Y$ *be sets of points in given Banach spaces X and Y, respectively. Let* $G = \sum_{i=1}^n \sum_{j=1}^m g_{i,j} x_i \otimes y_j$ *be an element in* $X \check{\otimes} Y$*, or equivalently an operator from* X^* *to Y. Denote*

$$\varepsilon_2(\{x_i\}_1^n) = \max\left\{\left\|\sum_{i=1}^n t_i x_i\right\|;\ \sum_{i=1}^n t_i^2 = 1\right\}.$$

Then,

$$E(\|G\|_{X\check{\otimes} Y}) \leqq \varepsilon_2(\{x_i\}_1^n) E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) + \varepsilon_2(\{y_j\}_1^m) E\left(\left\|\sum_{i=1}^n g_i x_i\right\|\right).$$

PROOF. We apply Theorem 2.1 with

$$\mathscr{E} = \{(x^*(x_1), \dots, x^*(x_n));\ \|x^*\| = 1,\ x^* \in X^*\}$$

and

$$\Theta = \{(y^*(y_1), \dots, y^*(y_m));\ \|y^*\| = 1,\ y^* \in Y^*\}.$$

Then obviously

$$E\left(\max_{\varepsilon \in \mathscr{E}} \max_{\theta \in \Theta} Y_{\varepsilon,\theta}\right) = E(\|G\|_{X\check{\otimes} Y})$$

and

$$E\left(\max_{\varepsilon \in \mathscr{E}} \max_{\theta \in \Theta} X_{\varepsilon,\theta}\right) = \varepsilon_2(\{x_i\}_1^n) E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) + \varepsilon_2(\{y_j\}_{j=1}^m) E\left(\left\|\sum_{i=1}^n g_i x_i\right\|\right). \quad \square$$

THEOREM 2.5. *Let Y be a Banach space,* $\{y_i\}_{i=1}^m \subset Y$*, and let* $\{e_i\}_{i=1}^n$ *be the unit vector basis of* l_2^n*. Let* $G(\omega) = \sum_{i=1}^n \sum_{j=1}^m g_{i,j}(\omega) e_i \otimes y_j$ *be a random operator from* l_2^n *to Y. Then*

$$E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) - \omega_n \varepsilon_2(\{y_j\}_1^m) \leqq E\left(\min_{\|x\|_2 = 1} \|G(x)\|\right) \leqq E(\|G\|)$$

(2.3)

$$\leqq E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) + \omega_n \varepsilon_2(\{y_j\}_1^m)$$

where

$$\omega_n = \sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right) \Big/ \Gamma\left(\frac{n}{2}\right) \qquad (\omega_n \leqq \sqrt{n} \text{ and } \omega_n n^{-1/2} \xrightarrow[n\to\infty]{} 1).$$

PROOF. A simple computation shows that $E(\sqrt{\sum_{i=1}^n g_i^2}) = \omega_n$ and hence

$$\omega_n \leqq \sqrt{E\left(\sum_{i=1}^n g_i^2\right)} = \sqrt{n}.$$

The right-hand side inequality follows immediately from Corollary 2.4 by setting $X = l_2^n$ and $x_i = e_i$, so $\varepsilon_2(\{x_i\}_1^n) = 1$ and $E(\|\sum_1^n g_i x_i\|) = E(\sqrt{\sum_{i=1}^n g_i^2}) = \omega_n$.

Let $\mathscr{E} = \{\varepsilon \in l_2^n;\ \|\varepsilon\|_2 = 1\}$ and $\Theta = \{(y^*(y_1), \dots, y^*(y_m));\ \|y^*\| = 1,\ y^* \in Y^*\}$. Then obviously in the notation of Theorem 2.1, $\theta_0 = \varepsilon_2(\{y_j\}_1^m)$,

$$\max_{\theta \in \Theta} \sum_{j=1}^m g_j \theta_j = \left\| \sum_{j=1}^m g_j y_j \right\|, \quad \min_{\varepsilon \in \mathscr{E}} \sum_{i=1}^n \varepsilon_i h_i = -\sqrt{\sum_{i=1}^n h_i^2},$$

thus we obtain that

$$E\left(\min_{\varepsilon \in \mathscr{E}} \max_{\theta \in \Theta} Y_{\varepsilon,\theta}\right) = E\left(\min_{\|x\|_2 = 1} \|G(x)\|\right)$$

and

$$E\left(\min_{\varepsilon \in \mathscr{E}} \max_{\theta \in \Theta} X_{\varepsilon,\theta}\right) = E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) - \varepsilon_2(\{y_j\}_1^m)\omega_n$$

so the left-hand side of (2.3) follows from Theorem 2.1. □

COROLLARY 2.6. *In the notation of Theorem 2.5, if*

$$E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) - \omega_n \varepsilon_2(\{y_j\}_{j=1}^m) > 0 \tag{2.4}$$

then Y contains an n-dimensional subspace Y_n such that

$$d(Y_n, l_2^n) \leqq \frac{E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) + \omega_n \varepsilon_2(\{y_j\}_1^m)}{E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) - \omega_n \varepsilon_2(\{y_j\}_1^m)}$$

$$\leqq \frac{E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) + \sqrt{n}\,\varepsilon_2(\{y_j\})}{E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) - \sqrt{n}\,\varepsilon_2(\{y_j\})}$$

PROOF. For each $\omega \in \Omega$, let

$$g(\omega) = \max_{\|x\|_2 = 1} \|G(\omega)x\| = \|G(\omega)\| \quad \text{and} \quad f(\omega) = \min_{\|x\|_2 = 1} \|G(\omega)x\|.$$

Obviously, $0 \leqq f(\omega) \leqq g(\omega)$ and since

$$0 < E\left(\left\|\sum_{j=1}^{m} g_j y_j\right\|\right) - \omega_n \varepsilon_2(\{y_j\}_{j=1}^{m}) \leqq E(f) \leqq E(g) \leqq E\left(\left\|\sum_{j=1}^{m} g_j y_j\right\|\right) + \omega_n \varepsilon_2(\{y_j\}_1^m)$$

it follows that there exists $\omega_0 \in \Omega$ for which

$$f(\omega_0)/g(\omega_0) \geqq \frac{E\left(\left\|\sum_{j=1}^{m} g_j y_j\right\|\right) - \omega_n \varepsilon_2(\{y_j\}_1^m)}{E\left(\left\|\sum_{j=1}^{m} g_j y_j\right\|\right) + \omega_n \varepsilon_2(\{y_j\}_1^m)}.$$

It follows now from the inequality

$$f(\omega_0)\| x \|_2 \leqq \| G(\omega_0) x \| \leqq g(\omega_0) \| x \|_2 \qquad (x \in l_2^n)$$

that if $Y_n = \operatorname{span}\{G(\omega_0)x\,;\, x \in l_2^n\}$, then $d(Y_n, l_2^n) \leqq g(\omega_0)/f(\omega_0)$. □

It is well known that

$$E\left(\left(\sum_{j=1}^{m} | g_j |^p\right)^{1/p}\right) = c_p(m) m^{1/p} \qquad \text{where } c_p(m) \underset{m\to\infty}{\longrightarrow} c_p\,(>0) \text{ for all } 0 < p < \infty,$$

and

$$E\left(\max_{1 \leqq j \leqq m} | g_j |\right) = c_\infty(m)\sqrt{\log m} \qquad \text{where } c_\infty(m) \underset{m\to\infty}{\longrightarrow} c_\infty(>0),$$

moreover,

$$c_1(m) = \sqrt{\frac{2}{\pi}}\,.$$

COROLLARY 2.7. *For any* $1 \leqq p \leqq \infty$, *and integers* $n \leqq m$, *if* $a(p, n, m) > 1$ *then* l_p^m *contains an* n*-dimensional subspace* Y_p *such that* $d(Y_p, l_2^n) \leqq b(p, n, m)$, *where*

(i) $a(p, n, m) = c_p^2(m) m n^{-1}$,

$$b(p, n, m) = (c_p(m) + \sqrt{n/m})/(c_p(m) - \sqrt{n/m}) \quad \textit{if } 1 \leqq p \leqq 2;$$

(ii) $a(p, n, m) = c_p^2(m) m^{2/p} n^{-1}$,

$$b(p, n, m) = (c_p(m) + n^{1/2} m^{-1/p})/(c_p(m) - n^{1/2} m^{-1/p}) \quad \textit{if } 2 \leqq p < \infty;$$

(iii) $a(p, n, m) = c_\infty^2(m)(\log m) n^{-1}$,

$$b(p, n, m) = (c_\infty(m)\sqrt{\log m} + \sqrt{n})/(c_\infty(m)\sqrt{\log m} - \sqrt{n}) \quad \textit{if } p = \infty.$$

PROOF. Let $\{y_j\}_{j=1}^m$ be the unit vector basis of l_p^m. Then

$$E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) = E\left(\left(\sum_{j=1}^m |g_j|^p\right)^{1/p}\right) = c_p(m) m^{1/p} \qquad \text{if } 1 \leqq p < \infty,$$

and

$$E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) = E\left(\max_j |g_j|\right) = c_\infty(m)\sqrt{\log m} \qquad \text{if } p = \infty.$$

In addition

$$\varepsilon_2(\{y_j\}_1^m) = \begin{cases} m^{1/p-1/2}, & \text{if } 1 \leqq p \leqq 2, \\ 1, & \text{if } p \geqq 2. \end{cases}$$

It follows that $a(p, n, m) > 1$ implies (2.4) and hence the conclusion of Corollary (2.6). We conclude the proof by noting that

$$b(p, n, m) = \frac{E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) + \sqrt{n}\,\varepsilon_2(\{y_j\}_1^m)}{E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) - \sqrt{n}\,\varepsilon_2(\{y_j\}_1^m)}. \qquad \square$$

The famous Dvoretzky's theorem was originally proved in [2], and many of its diverse and important applications were developed in [5]. We shall now show that the "proper" choice of the sequence $\{y_j\}_{j=1}^m$ in Corollary 2.6 proves this theorem with a sharper lower bound for N:

THEOREM 2.8. *There exists a constant $c > 0$ such that for any $1 > \varepsilon > 0$ and integer $n > 1$, if N is an integer satisfying $N \geqq \exp(cn\varepsilon^{-2})$, then any N-dimensional Banach space Y contains an n-dimensional subspace Y_n for which $d(Y_n, l_2^n) \leqq (1+\varepsilon)/(1-\varepsilon)$.*

PROOF. By the Dvoretzky–Rogers' theorem [3], if Y is an N-dimensional Banach space, there exists an inner product norm on Y denoted by $\|\cdot\|_2$ and there is a sequence $\{y_j\}_{j=1}^N \subset Y$ such that:

(I) $1 = \|y_j\|_2 = \|y_j\|_Y = \|y_j\|_{Y^*}$ $(1 \leqq j \leqq N)$;

(II) $\|\sum_{j=1}^m t_j y_j\| \leqq \sqrt{1 + m(m-1)/N}(\sum_{j=1}^m t_j^2)^{1/2}$ for every $1 \leqq m \leqq N$ and for all real numbers $\{t_j\}_1^m$;

(III) there exists an orthonormal basis $\{u_j\}_{j=1}^N$ such that $y_k = \sum_{i=1}^k y_{k,i} u_i$, where

$$\sum_{i=1}^{k-1} y_{k,i}^2 = 1 - y_{k,k}^2 \leqq \frac{k-1}{N} \qquad \text{for all } 1 \leqq k \leqq N.$$

For such a choice of $\{y_j\}_{j=1}^N$ we obtain from (II) that

$$\varepsilon_2(\{y_j\}_1^m) \leqq \sqrt{1 + m(m-1)/N} \leqq \sqrt{1 + m^2/N},$$

and by (I)

$$E\left(\left\|\sum_{j=1}^m g_j y_j\right\|\right) \geqq E\left(\max_{1 \leqq j \leqq m} |g_j| \, \|y_j\|\right) \geqq c_1 \sqrt{\log m}$$

where c_1 is a positive constant. Substituting these inequalities in Corollaary 2.6, we obtain that if

$$c_1 \sqrt{\log m} - \sqrt{1 + m^2/N}\sqrt{n} > 0$$

then

$$d(Y_n, l_2^n) \leqq \frac{c_1 \sqrt{\log m} + \sqrt{n}\sqrt{1 + m^2/N}}{c_1 \sqrt{\log m} - \sqrt{n}\sqrt{1 + m^2/N}}.$$

The r.h.s. of the last inequality is smaller than $(1+\varepsilon)/(1-\varepsilon)$ if

$$\varepsilon^2 \geqq n(1 + m^2/N)/c_1^2 \log m.$$

Picking $m = [N^{1/2}]$ it is easy to see that there exists a universal constant $c > 0$ for which the last inequality is satisfied provided $\log n \geqq cn\varepsilon^{-2}$. □

The above proof can be generalized to yield the general Dvoretzky's theorem about arbitrary convex bodies in R^N which in our formulation states:

Theorem 2.9. *Given any $0 < \varepsilon < 1$ and integer n, there exists an integer $N = N(\varepsilon, n)$, so that if B is any convex body in R^N with non-empty interior, and E is the ellipsoid of maximal volume contained in B, and if the origin of R^N is chosen to be the center of E, then there exists $\lambda > 0$ and an n-dimensional subspace H of R^N for which*

$$\lambda(E \cap H) \subseteq B \cap H \subseteq \lambda\left(\frac{1+\varepsilon}{1-\varepsilon}\right)(E \cap H).$$

The proof of this theorem is carried out on similar lines. One needs to replace the norm of the Banach space Y by the Minkowsky functional $p(.)$ associated with the convex set B, and to verify that a suitable Dvoretzky–Rogers' lemma can be formulated for B, that is, for the space $(R^N, p(.))$ which replaces Y. We leave the details to the interested reader.

REMARK. The lower bound for N, $\exp(cn\varepsilon^{-2})$, of Theorem 2.8 above is an improvement on the one obtained in [5] which, by using ε-nets to cover the unit sphere of l_2^n and the isoperimetric inequality, gives the lower bound

$$N > \exp\left(cn\varepsilon^{-2}\log\left(\frac{2}{\varepsilon}+1\right)\right).$$

Using Theorem 1.1 and 2.3, we can derive some inequalities which generalize Theorem 2.5, but, of course, these cannot be quite as sharp when applied for the special case which was considered there.

As before, denote by $\{g_{i,j}\}$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$, $\{h_i\}_{i=1}^n$, $\{g_j\}_{j=1}^m$ and g, independent sets of orthonormal Gaussian r.v.s.

THEOREM 2.10. *Let $\{x_i^*\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ be finite sets of points in Banach spaces E^* and F, respectively, where E^* is the dual space to E. Let*

$$G = \sum_{i=1}^n \sum_{j=1}^m g_{i,j} x_i^* \otimes y_j$$

be the Gaussian operator from E to F. If

$$X = \min_{\|x\|=1}\left(\sum_{i=1}^n |x_i^*(x)|^2\right)^{1/2}\left[\left\|\sum_{j=1}^m g_j y_j\right\| - \varepsilon_2(\{y_j\}_1^m)\left(\sum_{i=1}^n h_i^2\right)^{1/2}\right],$$

$$Y = \min_{\|x\|=1}\|G(x)\| + \varepsilon_2(\{x_i^*\}_1^n)\varepsilon_2(\{y_j\}_1^m)|g|,$$

$$Z = \|G\| - \varepsilon_2(\{x_i^*\}_1^n)\varepsilon_2(\{y_j\}_1^m)|g|,$$

$$W = \varepsilon_2(\{x_i^*\}_1^n)\left\|\sum_{j=1}^m g_j y_j\right\| + \varepsilon_2(\{y_j\}_1^m)\left\|\sum_{i=1}^n h_i x_i^*\right\|,$$

then for all real scalars λ

$$P(X \geqq \lambda) \leqq P(Y \geqq \lambda) \quad \text{and} \quad P(Z \geqq \lambda) \leqq P(W \geqq \lambda).$$

PROOF. Let $\mathscr{E} = \{(x_1^*(x), \dots, x_n^*(x));\ x \in E,\ \|x\| = 1\} \subset R^n$ and

$$\Theta = \{(y^*(y_1), \dots, y^*(y_m));\ y^* \in F^*, \|y^*\| = 1\} \subset R^m.$$

Now define $X_{\varepsilon,\theta}$ and $Z_{\varepsilon,\theta}$ as in Theorem 2.3. Inequalities (I) and (II) of Theorem 2.3 imply by Theorem 1.1 that

$$P\left(\min_\varepsilon \max_\theta X_{\varepsilon,\theta} \geqq \lambda\right) \leqq P\left(\min_\varepsilon \max_\theta Z_{\varepsilon,\theta} \geqq \lambda\right)$$

and

$$P\left(\max_{\varepsilon,\theta} Z_{\varepsilon,\theta} \geqq \lambda\right) \leqq P\left(\max_{\varepsilon,\theta} X_{\varepsilon,\theta} \geqq \lambda\right).$$

But clearly,

$$\min_{\varepsilon} \max_{\theta} X_{\varepsilon,\theta} \geqq X, \quad \min_{\varepsilon} \max_{\theta} Z_{\varepsilon,\theta} \leqq Y, \quad \max_{\varepsilon,\theta} Z_{\varepsilon,\theta} \geqq Z \quad \text{and} \quad \max_{\varepsilon,\theta} X_{\varepsilon,\theta} \leqq W,$$

from which the conclusion follows. □

COROLLARY 2.11. *If $f(t)$ is increasing in $(-\infty,\infty)$, then $E(f(X)) \leqq E(f(Y))$ and $E(f(Z)) \leqq E(f(W))$.*

REMARKS. (a) In particular, taking $f(t) = t$, and $E = l_2^n$, we obtain from Corollary 2.11 an inequality similar to inequality (2.3) of Theorem 2.5 (which, of course, is not quite as sharp). However, we note Dvoretzky's theorem can be similarly deduced from this weaker inequality as well.

(b) In Theorem 2.3, the sets $\mathscr{E}$ and Θ were both finite, whereas in Theorem 2.10 they are not. This does not cause any real difficulty in the proof of Theorem 2.10, because we can always approximate $\mathscr{E}$ and Θ by their finite subsets.

Recall now the notions of type and cotype. The type p (cotype q) on m vectors of a Banach space Y, denoted by $T_m^{(p)}(Y)$ (resp., $c_m^{(q)}(Y)$), is the least constant α such that for every subset $\{y_i\}_{i=1}^m \subset Y$

$$E\left(\left\|\sum_{i=1}^m g_i y_i\right\|\right) \leqq \alpha\left(\sum_{i=1}^m \|y_i\|^p\right)^{1/p} \quad \left(\text{resp., } \left(\sum_{i=1}^m \|y_i\|^q\right)^{1/q} \leqq \alpha E\left(\left\|\sum_{i=1}^m g_i y_i\right\|\right)\right).$$

The type p (cotype q) constant of Y is

$$T^{(p)}(Y) = \sup_m T_m^{(p)}(Y) \quad \left(\text{resp., } c^{(q)}(Y) = \sup_m c_m^{(q)}(Y)\right).$$

To make sense we must have $1 \leqq p \leqq 2 \leqq q \leqq \infty$. We shall next show that if Y has cotype q, $\dim Y \geqq m$, where m is "big" compared to $n^{q/2}$, then Y contains an n-dimensional subspace "close" to l_2^n. This result can be found in [5] as well, but the numerical estimate seems to be new.

COROLLARY 2.12. *If* $\dim Y = m$, *where* $2c^{(q)}(Y)m^{-1/q}\omega_n < 1$, *then Y contains an n-dimensional subspace Z for which*

$$d(Z, l_2^n) \leqq (1 + 2c^{(q)}(Y)m^{-1/q}\omega_n)/(1 - 2c^{(q)}(Y)m^{-1/q}\omega_n)$$

where

$$\omega_n = \sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right) \Big/ \Gamma\left(\frac{n}{2}\right).$$

PROOF. Let B_2 be the ellipsoid of maximal volume contained in the unit ball of Y. Without loss of generality we may assume that $B_2 = \{(t_i)_{i=1}^m;\ \Sigma t_i^2 \leqq 1\}$. By [7], there exists a sequence

$$\{y_r\}_{r=1}^s \subset Y, \qquad m \leqq s \leqq \frac{m(m+1)}{2},$$

and positive scalars λ_r such that

(1) $\|y_r\|_Y = \|y_r\|_2 = 1$ for all $1 \leqq r \leqq s$,

(2) $\Sigma_1^s \lambda_r = m$,

(3) $y = \Sigma_{r=1}^s \lambda_r (y_r, y) y_r$ for all vectors y.

Since $\|.\|_Y \leqq \|.\|_2$ it follows that

$$\varepsilon_2(\{\sqrt{\lambda_r} y_r\}_{r=1}^s) = \sup\left\{\left(\sum_{r=1}^s \lambda_r (y_r, y)^2\right)^{1/2};\ \|y\| = 1,\ y \in Y^*\right\}$$

$$= \sup\{\|y\|_2;\ \|y\| = 1,\ y \in Y^*\} \leqq 1,$$

so using the inequality $2C_m^{(2)}(Y) \geqq C^{(2)}(Y)$ [9], together with $C_m^{(q)}(Y) m^{1/2-1/q} \geqq C_m^{(2)}(Y)$, we obtain that

$$E\left(\left\|\sum_{r=1}^s \sqrt{\lambda_r} g_r y_r\right\|\right) \geqq (C^{(2)}(Y))^{-1}\left(\sum_{r=1}^s \lambda_r\right)^{1/2} \geqq \sqrt{m}(2C_m^{(2)}(Y))^{-1}$$

$$\geqq m^{1/q}/2C_m^{(q)}(Y) \geqq m^{1/q}/2C^{(q)}(Y),$$

hence if we replace in Corollary 2.6 the sequence $\{y_i\}$ by $\{\sqrt{\lambda_r} y_r\}_{r=1}^s$ the required estimate will follow.

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